Spatiotemporal stability of one-way open coupled nonlinear systems

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The present paper investigates the spatiotemporal stability of homogeneous solutions in one-way open coupled nonlinear systems. We show that the H_∞ norm concept, which has been used as an important index in the field of robust control theory, allows us to grasp the mechanism of spatial instability in coupled systems. Spatial instability occurs only when the H_∞ norm of the transfer function of each site is greater than 1. It is shown that numerical simulations for one-way open coupled double scroll circuits are in good agreement with our theoretical results.

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I. INTRODUCTION

Chaotic behavior and bifurcations in nonlinear systems have been widely studied [1,2]. Several types of bifurcation have been investigated from many points of view [3]. Bifurcations occur in continuous-time nonlinear systems when at least one of the eigenvalues of the Jacobi matrix around a fixed point intersects the imaginary axis in the complex plane. The bifurcation type depends on the crossing points where the eigenvalues intersect the imaginary axis. Furthermore, the location of the eigenvalues is a unique criterion for classification of the fixed-point type. We may, therefore, reasonably conclude that the fixed points of nonlinear systems have been examined using the eigenvalues of Jacobi matrix. Recently, the dynamics of spatially extended nonlinear systems has gained much attention in the field of nonlinear science. The dynamics of extended systems is too complicated to analyze the bifurcations and stability theoretically. Thus most studies on bifurcations and stability have used computer simulations instead of theoretical approaches.

Coupled map lattices (CMLs) have been particularly investigated by many researchers, since they exhibit a wide variety of complex spatiotemporal behavior [4]. CMLs have discrete time, discrete space, and continuous-state variables. The one-way open CML is well known as a typical open flow model [5–8]. Kaneko discovered that spatial instability occurs in the one-way open CML [5], and it has been examined in detail by numerical simulation [6,7]. Yamaguchi investigated the instability and derived the bifurcation conditions [9]. Konishi, Kokame, and Hirata examined the mechanism of the instability by using the H_∞ norm concept of control theory [10]. Johnson, Löcher, and Hunt found a spatial period-doubling bifurcation in one-way open coupled diode resonator circuits [11]. However, it remains an open question how to clarity the mechanism of spatial instability in one-way open coupled *continuous-time* nonlinear systems.

The present paper investigates the spatiotemporal stability of a homogeneous solution in one-way open coupled continuous-time nonlinear systems. The spatiotemporal stability has two criteria: the eigenvalues of the Jacobi matrix (i.e., the poles of a transfer function) and the H_∞ norm of a transfer function. In order to confirm our theoretical results, we simulate the one-way open coupled double scroll circuits proposed in [12].

II. ONE-WAY OPEN COUPLED NONLINEAR SYSTEMS

Let us consider a one-way open coupled nonlinear system

$$\dot{\mathbf{x}}(i)=\mathbf{f}(x(i))+\mathbf{b}\{x_1(i-1)-x_1(i)\} \quad (i=1,2,...,N), \tag{2.1}$$

where $\mathbf{x}(i):=[x_1(i)\,x_2(i)\cdots x_n(i)]^T\in\mathbb{R}^n$ is the n-dimensional system state of the ith lattice site and $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$ is an n-dimensional nonlinear function. The coupling vector is $\mathbf{b}=[\varepsilon 0\cdots 0]^T\in\mathbb{R}^n$, where ε is the coupling strength. Figure 1 illustrates a one-way open coupled nonlinear system. If the upper boundary $x_1(0)$ is fixed at x_{f1}, a homogeneous solution of system (2.1) is given by

$$[\mathbf{x}(1)\ \ \mathbf{x}(2)\ \ \cdots\ \ \mathbf{x}(N)]=[\mathbf{x}_f\ \ \mathbf{x}_f\ \ \cdots\ \ \mathbf{x}_f], \tag{2.2}$$

where

$$\mathbf{x}_f=[x_{f1}\ \ x_{f2}\ \ \cdots\ \ x_{fn}]^T\in\mathbb{R}^n. \tag{2.3}$$

The fixed point $\mathbf{x}_f$ of each site satisfies $\mathbf{f}(\mathbf{x}_f)=\mathbf{0}$. We shall consider the spatiotemporal stability of homogenous solution (2.2) in Sec. III.

III. STABILITY ANALYSIS

For simplicity, we focus on the dynamics of the ith lattice site. Assume that the dynamics of the upper sites [i.e., the 1, 2, ..., $(i-2)$th lattice sites] has already converged to the homogeneous solution [i.e., $\mathbf{x}(m)=\mathbf{x}_f$, $m=1,2,...,(i-2)$]. If the $(i-1)$th and ith lattice site states $\mathbf{x}(i-1)$ and $\mathbf{x}(i)$ are in the neighborhood of the fixed point $\mathbf{x}_f$, then the dynamics of the ith lattice site is governed by

$$\dot{\mathbf{y}}(i)=\mathbf{A}\mathbf{y}(i)+\mathbf{b}y_1(i-1),$$

$$y_1(i)=\mathbf{c}\mathbf{y}(i), \tag{3.1}$$

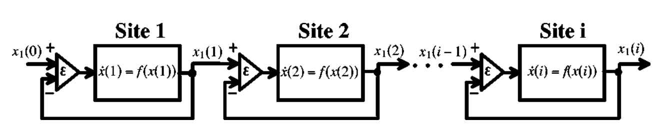

FIG. 1. Block diagram of one-way open coupled nonlinear system.

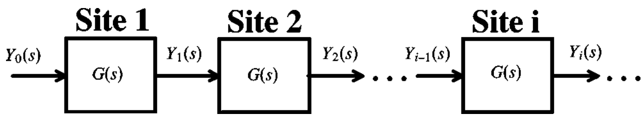

FIG. 2. Frequency domain block diagram around the homogeneous solution.

where

$$\mathbf{y}(i) := \mathbf{x}(i) - \mathbf{x}_f = [y_1(i) \;\; y_2(i) \;\; \cdots \;\; y_n(i)]^T,$$

$$\mathbf{A} := \left.\frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_f} + \begin{bmatrix} -\varepsilon & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \mathbf{c} := [1 \;\; 0 \;\; \cdots \;\; 0].$$

Note that the input and output of system (3.1) are $y_1(i-1)$ and $y_1(i)$, respectively. In the frequency domain, the input-output relation of system (3.1) can be described as

$$Y_i(s) = G(s) Y_{i-1}(s), \tag{3.2}$$

where

$$Y_i(s) := \mathcal{L}[y_1(i)], \quad Y_{i-1}(s) := \mathcal{L}[y_1(i-1)],$$

$$G(s) = N(s)/D(s).$$

$\mathcal{L}$ denotes the Laplace transfer function. $N(s)$ and $D(s)$ are polynomials. The derivation of Eq. (3.2) is given in the Appendix. If the $(i-1)$th lattice site is fixed at $\mathbf{x}(i-1) = \mathbf{x}_f$, the ith lattice state $\mathbf{x}(i)$ converges on $\mathbf{x}_f$ only when $G(s)$ is stable (i.e., all the eigenvalues of the system matrix $\mathbf{A}$ lie in the open left half of the complex plane). Figure 2 sketches the block diagram of the coupled linear system (3.2). Let us assume that the $(i-1)$th lattice site is disturbed as $x_1(i-1) = x_{f1} + y_1(i-1)$, where the small disturbance is set as $y_1(i-1) = \delta_{i-1} \sin \omega t$. Then the ith lattice site is given by $x_1(i) = x_{f1} + y_1(i)$, where $y_1(i) = \delta_i \sin(\omega t + \phi)$. The amplitude ratio of δ_i to δ_{i-1} can be described as

$$\frac{\delta_i}{\delta_{i-1}} = |G(j\omega)|.$$

Suppose that all site states converge to the fixed point $\mathbf{x}_f$. The small sinusoidal disturbance $\delta_p \sin \omega t$ is added only to the pth lattice site; the influence of the disturbance can be observed as $\delta_q \sin(\omega t + \varphi)$ at the qth lattice site. The amplitude ratio of δ_q to δ_p is given by

$$\frac{\delta_q}{\delta_p} = |G(j\omega)|^{q-p}, \tag{3.3}$$

where $q > p$. Now consider the following three case: (i) $|G(j\omega)| < 1$, (ii) $|G(j\omega)| = 1$, (iii) $|G(j\omega)| > 1$. For case (i), the amplitude ratio decreases about exponentially with $(q-p)$ [see Eq. (3.3)]. Therefore, if a sinusoidal signal with angular frequency ω is added to an upper lattice site, the signal has little effect on lower sites. For case (iii), the amplitude ratio increases about exponentially with $(q-p)$. If a tiny sinusoidal signal with angular frequency ω is added to an upper site, the lower sites are significantly influenced by the signal. For case (ii), the influence of the added signal at the upper sites is constant for all the sites.

For real systems with external noise or for numerical systems with round-off error on computers, we have to investigate spatial robustness for all $\omega \in \mathbb{R}$. In order to simplify the discussions below we introduce the H_∞ norm concept [13].

Definition 1. Assume that a transfer function $G(s)$ is stable. The H_∞ norm of the transfer function $G(s)$ is given by [13]

$$\|G(s)\|_\infty := \sup_{\omega \in \mathbb{R}} |G(j\omega)|.$$

Roughly speaking, the H_∞ norm is the peak gain of the bode diagram of $G(s)$. We note that the H_∞ norm concept is useful in defining spatial stability; hence, we give a simple definition of the spatiotemporal stability of homogeneous solution (2.2) in one-way open coupled nonlinear systems.

Definition 2. The spatiotemporal stability of homogeneous solution (2.2) in one-way open coupled nonlinear system (2.1) is classified into the following three types: (i) If $G(s)$ is unstable (i.e., at least one eigenvalue of system matrix $\mathbf{A}$ is in the open right half of the complex plane), it is temporally unstable (TU). (ii) If $G(s)$ is stable and $\|G(s)\|_\infty < 1$, it is temporally spatially stable (TSS). (iii) If $G(s)$ is stable and $\|G(s)\|_\infty > 1$, it is temporally stable and spatially unstable (TSSU).

This definition can be regarded as a continuous-time version of the spatiotemporal stability introduced in Ref. [10]

If solution (2.2) is TU, we can observe the oscillation for all sites. On the contrary, if it is TSS, no sites oscillate. If it is TSSU, some upper sites never oscillate. However, the noise at the upper sites induces oscillation in the lower sites. This is because a tiny external noise or a round-off error on a computer at an upper site significantly disturbs the lower sites. Hence, the lower site states $\mathbf{x}(i)$ cannot keep staying on $\mathbf{x}_f$.

In order to check the stability of homogeneous solution (2.2), we have to estimate the eigenvalues of matrix $\mathbf{A}$ and the H_∞ norm of $G(s)$ in advance. For coupled high-dimensional nonlinear systems, the analytical estimation of the eigenvalues and the norm is not easy; however, a software package [14] allows us to obtain a numerical estimation of the eigenvalues and norm by simple commands. For example, the eigenvalues and the norm can be estimated by the commands SPOL and NORMINF, respectively.

IV. ONE-WAY OPEN COUPLED DOUBLE SCROLL CIRCUITS

We shall consider the one-way open coupled double scroll circuits proposed by Kapitaniak, Chua, and Zhong [12] as a numerical example. They reported experimental observation of hyperchaotic attractors in the coupled circuits. In our paper, we investigate the spatiotemporal stability of the homogeneous solution of the coupled circuits. The simple dimensionless state equation of the circuits [12] can be written as

$$\dot{x}_1(i) = -x_1(i) + x_2(i) + x_3(i) + \varepsilon\{x_1(i-1) - x_1(i)\},$$

$$\dot{x}_2(i) = \alpha\{x_1(i) - x_2(i) - h(x_2(i))\}, \tag{4.1}$$

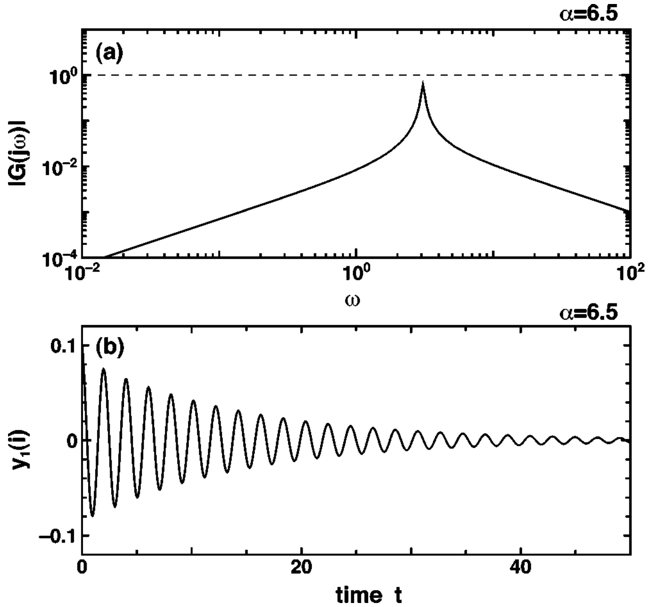

FIG. 3. Bode diagram (a) and impulse response (b) of $G(s)$ for $\alpha=6.5$.

$$\dot{x}_3(i)=-\beta x_1(i),$$

for $i=1,2,...,N$. The nonlinear function $h(x_2(i))$ is

$$h(x_2(i))=\begin{cases} bx_2(i)+a-b & \text{if } x_2(i)\geqslant 1 \\ ax_2(i) & \text{if } |x_2(i)|\leqslant 1 \\ bx_2(i)-a+b & \text{if } x_2(i)\leqslant -1 \end{cases}.$$

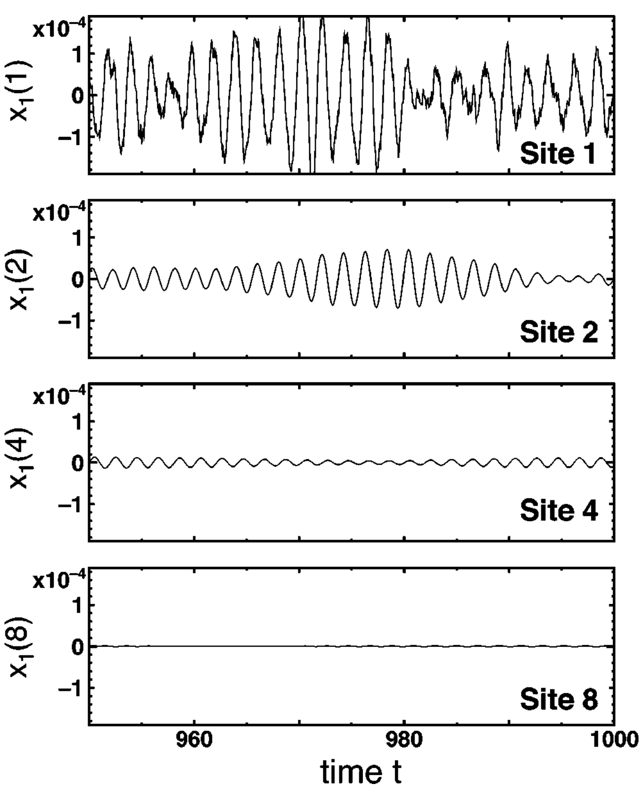

FIG. 4. Behavior of four sites for $\alpha=6.5$

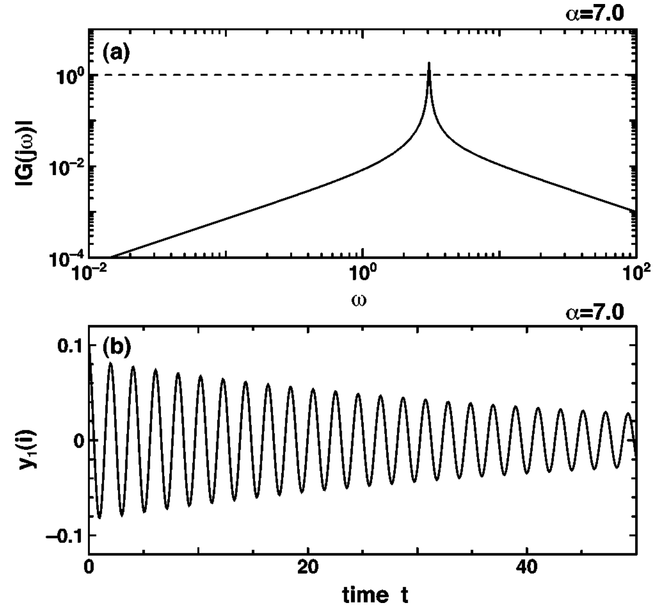

FIG. 5. Bode diagram (a) and impulse response (b) of $G(s)$ for $\alpha=7.0$.

Our analysis will be based on the dynamics (4.1). Each circuit has three fixed points:

$$\mathbf{x}_f^{(+)}=\begin{bmatrix} 0 \\ +\lambda \\ -\lambda \end{bmatrix}, \quad \mathbf{x}_f^{(0)}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_f^{(-)}=\begin{bmatrix} 0 \\ -\lambda \\ +\lambda \end{bmatrix},$$

where $\lambda=(b-a)/(b+1)$. Let us consider the stability of the following homogeneous solution:

$$[\mathbf{x}(1) \ \ \mathbf{x}(2) \ \ \cdots \ \ \mathbf{x}(N)]^T=[\mathbf{x}_f^{(+)} \ \ \mathbf{x}_f^{(+)} \ \ \cdots \ \ \mathbf{x}_f^{(+)}]^T. \tag{4.2}$$

The upper boundary $x_1(0)$ is set at

$$x_1(0)=0+\eta\sigma(t),$$

where $-1\leqslant\sigma(t)\leqslant+1$ is the uniform random noise and η is the small noise level. The error dynamics around homogeneous solution (4.2) is

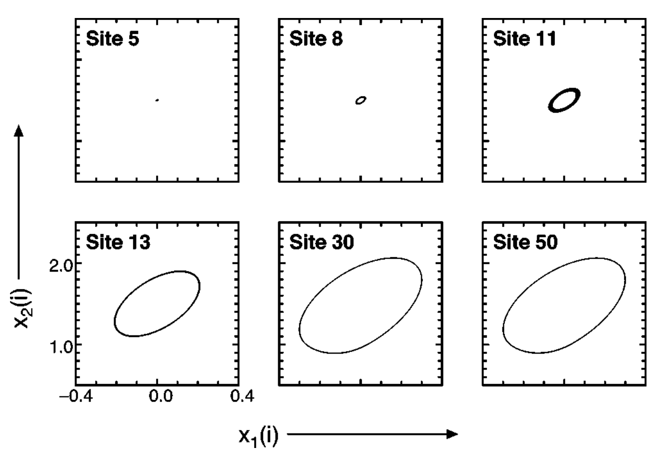

FIG. 6. Trajectories in the phase plane for $\alpha=7.0$.

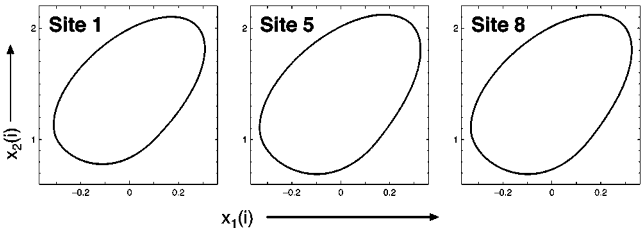

FIG. 7. Trajectories in the phase plane for $\alpha=7.5$.

$$\begin{bmatrix}\dot{y}_1(i)\\ \dot{y}_2(i)\\ \dot{y}_3(i)\end{bmatrix}=\begin{bmatrix}-1-\varepsilon & 1 & 1\\ \alpha & -\alpha(b+1) & 0\\ -\beta & 0 & 0\end{bmatrix}\begin{bmatrix}y_1(i)\\ y_2(i)\\ y_3(i)\end{bmatrix}+\begin{bmatrix}\varepsilon\\ 0\\ 0\end{bmatrix}y_1(i-1),$$

$$y_1(i)=[1\quad 0\quad 0]\begin{bmatrix}y_1(i)\\ y_2(i)\\ y_3(i)\end{bmatrix}.$$

We note that this dynamics corresponds to system (3.1). The transfer function from $y_1(i-1)$ to $y_1(i)$ is given by $G(s)=N(s)/D(s)$. The polynomials are $N(s)=\varepsilon s^2+\alpha\varepsilon(1+b)s$ and $D(s)=s^3+\rho_3 s^2+\rho_2 s+\rho_1$, where $\rho_1=\alpha\beta(1+b)$, $\rho_2=\alpha b+\beta+\alpha\varepsilon(1+b)$, $\rho_3=1+\varepsilon+\alpha(1+b)$. The Routh-Hurwitz stability test [15] allows us to obtain the necessary and sufficient condition for $G(s)$ to be stable, that is,

$$\rho_1>0,\quad \rho_3>0,\quad \rho_2\rho_3-\rho_1>0. \tag{4.3}$$

Our example employs the famous parameters [16]

$$a=-\tfrac{8}{7},\quad b=-\tfrac{5}{7},\quad \beta=14+\tfrac{2}{7}.$$

The noise level at the upper boundary is set as $\eta=10^{-2}$. The coupling strength is fixed at $\varepsilon=0.1$. Now we shall check the stability of homogeneous solution (4.2) for three cases: $\alpha=6.5$, 7.0, and 7.5.

For the case $\alpha=6.5$, we derive the polynomials $N(s)=0.1s^2+0.1857s$ and $D(s)=s^3+2.9571s^2+9.8286s+26.5306$. It is confirmed that ρ_1, ρ_2, and ρ_3 satisfy stability condition (4.3). We estimate $\|G(s)\|_\infty=0.6029$ by the software [14], then we note that the solution is TSS from Definition 2. In order to confirm the stability obtained above, we show the bode diagram and the impulse response of $G(s)$ in Fig. 3. Since the peak gain of $G(s)$ is less than 1 and the impulse response of $G(s)$ is stable, we see that $\|G(s)\|_\infty$ is less than 1 and $G(s)$ is stable. Figure 4 illustrates the behavior of the first, second, fourth, and eighth sites. The first site $x_1(1)$ is disturbed by the noise of the upper boundary. The influence of the disturbance decreases about exponentially with the site number. At the eighth site, we observe little influence of the upper boundary noise.

For the case $\alpha=7.0$, we derive $N(s)=0.1s^2+0.2s$ and $D(s)=s^3+3.1s^2+9.4857s+28.5714$. The coefficients ρ_1, ρ_2, and ρ_3 satisfy stability condition (4.3). We estimate $\|G(s)\|_\infty=1.9094$; then we see that the solution is TSSU. Figure 5 shows the bode diagram and the impulse response of $G(s)$. It is confirmed that $\|G(s)\|_\infty$ is greater than 1 and $G(s)$ is stable. The trajectories in the phase plane for sites 5, 8, 11, 13, 30, and 50 are shown in Fig. 6. It can be seen that the tiny noise at the upper boundary significantly disturbs lower sites, and causes oscillation in the lower sites.

For the case $\alpha=7.5$, the polynomials are $N(s)=0.1s^2+0.2143s$ and $D(s)=s^3+3.2429s^2+9.1429s+30.6122$. The coefficients ρ_1, ρ_2, and ρ_3 do not satisfy stability condition (4.3), and we see that the solution is TU. The trajectories in the phase plane for the first, fifth, and eighth sites are shown in Fig. 7. As one can see, the upper edge site (i.e., the first site) oscillates with large amplitude, while we cannot observe an upper edge oscillation in the TSS and TSSU regimes.

V. CONCLUSIONS

We have investigated the spatiotemporal stability of a homogeneous solution in one-way open coupled continuous-time nonlinear systems. The main result obtained is as follows: spatial instability in one-way open coupled continuous-time nonlinear systems is clarified by the H_∞ norm of each site transfer function. It should be noted that our theoretical results do not depend on the system size (i.e., number of sites) or the dimension of sites.

APPENDIX

The Laplace transform of system (3.1) is given by

$$s\mathbf{Y}(s)=\mathbf{A}\mathbf{Y}(s)+\mathbf{b}Y_{i-1}(s), \tag{A1a}$$

$$Y_i(s)=\mathbf{c}\mathbf{Y}(s), \tag{A1b}$$

where $\mathbf{Y}(s)=\mathcal{L}[\mathbf{y}(i)]$. Equation (A1a) can be described as

$$\mathbf{Y}(s)=(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{b}Y_i(s). \tag{A2}$$

From Eqs. (A1b) and (A2), we can obtain system (3.2), where $G(s)=\mathbf{c}(s\mathbf{I}_n-\mathbf{A})^{-1}\mathbf{b}$.

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